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LITERATURE REVIEW OF CENSORING SCHEMES IN COMPETING RISKS MODELS

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ABSTRACT

Censoring schemes play an important role in survival analysis, particularly within the framework of competing risks models. These models are essential in analyzing time-to-event data where multiple potential failure causes exist, and not all events are fully observed due to censoring. Various types of censoring schemes which include Type-I, Type-II, hybrid, and the progressive censoring, were developed to handle incomplete data, providing robust methodologies for estimating survival functions and hazard rates. This review explores the theoretical underpinnings and practical applications of these censoring schemes in competing risks models. Emphasis is placed on recent advancements in the field, highlighting their significance in a range of disciplines, from reliability engineering to clinical trials. This study aims to provide a comprehensive overview of the different censoring schemes and their impact in the analysis of competing risks, offering insights into their strengths and limitations. By synthesizing the latest research, this paper contributes to the ongoing discourse on how to improve the accuracy and applicability of reliability analysis in the presence of competing risks data.

INTRODUCTION

Censoring schemes play a pivotal role in survival analysis, particularly in the study of competing risks models, where multiple potential causes of failure are involved, and not all events are fully observed due to incomplete data. These models are essential for accurately analyzing time-to-event data across various fields, such as biomedical research, reliability engineering, and actuarial science. In scenarios where multiple failure types can occur, censoring mechanisms allow researchers to handle incomplete or partially observed data efficiently, ensuring robust estimation of survival functions, hazard rates, and other important parameters.

Traditionally, censoring schemes are classified into several types, including Type-I, Type-II, hybrid, and progressive censoring. Type-I censoring occurs when a study is terminated after a pre-specified time, regardless of the number of failures observed. Type-II censoring, on the other hand,

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terminates when a pre-defined number of failures are observed. Hybrid censoring combines features of both Type-I and Type-II, while progressive censoring removes units progressively over time as failures are observed, offering more flexibility in handling incomplete data.

Competing risks models become more complex when combined with these censoring schemes, especially in studies where multiple failure modes must be accounted for simultaneously. These models allow for a better understanding of how different factors contribute to failures under various conditions. Recent advancements in statistical methods have improved the analysis of censored data within competing risks frameworks, particularly through the use of Bayesian and classical estimation techniques.

This paper provides a comprehensive review of censoring schemes in the context of competing risks models, exploring the theoretical underpinnings, methodological developments, and practical applications across disciplines. By examining recent research, this paper highlights how various censoring schemes enhance the accuracy and applicability of reliability analysis in the presence of competing risks. Through this review, we aim to offer insights into the strengths and limitations of these censoring methods and their potential for further development in survival analysis.

The relationship between the hazard and the survival functions

The hazard function and the survival function are two fundamental concepts in survival analysis that describe different aspects of the time-to-event data, particularly in the context of reliability studies, clinical trials, and risk analysis.

the survival function S(t) can be derived from the hazard function h(t), and vice versa. The relationship is as follows:

From Hazard Function to Survival Function

The survival function is related to the hazard function by the cumulative hazard function H(t), which is the integral of the hazard function over time. The survival function can be written as:

$$S(t) = exp(-H(t)) \quad [1]$$

Where H(t) is the cumulative hazard function, defined as:

$$H(t) = \int_0^t h(u)du \quad [2]$$

List of abbreviations

T-I CS	Type-I Censoring Scheme
HIV	Human Immunodeficiency Virus
AT-II Prog CS	Adaptive Type-II Progressive Censoring Scheme
ProgT-II CS	Progressive Type-II Censoring Scheme
T-II CS	Type-II Censoring Scheme
T-II ProgH CS	Type-II Progressive Censoring Scheme

Type-I and Type-II Censoring in Competing Risks Models

Type-I Censoring is described as: let n be the number of independent units that were placed on the life test. Before the start of the life-test, the experimenter specifies an exact time T, where the experiment stops when it reaches that exact time. suppose that the failure of lifetimes $t_1, t_2, \ldots, t_m$ are observed and failed by T only, suppose m takes any random value from 0 into n, which can be defined as random variable. While the remaining $(n-m)$ were censored and their lifetimes is known to be greater than T. The *pdf* of a T-I CS sample $t = t_{1:n}, t_{2:n}, \ldots, t_{m:n}$ takes the form as

$$f(t_1, t_2, \ldots, t_m) = \frac{n!}{(n-m)!}\left[\prod_{i=1}^{m} f(t_{i:n})\right][1 - F(T)]^{n-m}, \quad [3]$$
$$0 < t_{1:n} < t_{2:n} < \ldots < t_{m:n} < T.$$

The control of the longevity of life test is considered as one of the most famous features when using T-I CS. However, the observed failures could be a very little number that leads to a maximum likelihood estimation procedure that becomes complicated.

Type-II Censoring (T-II CS): suppose that n independent units are put on the life test. Before starting a life test, the experimenter specifies a number m (known as a number of the observed failures during the experiment), the experiment is ended when the m^{th} failure happen. Consider the failure lifetimes $x_1, x_2, ..., x_m, (m < n)$ are observed, but the time spent on the experiment is a random variable. While the remaining surviving units $(n-m)$ are withdrawn from the experiment. The joint pdf of a T-II CS sample $x = x_{1:n}, x_{2:n}, ..., x_{m:n}$ can be defined as

$$f(x_1, x_2, \ldots, x_m) = \frac{n!}{(n-m)!}\left[\prod_{i=1}^{m} f(x_{i:n})\right]\left[1 - F(x_{m:n})\right]^{n-m}, \quad [4]$$
$$0 < x_{1:n} < x_{2:n} < \ldots < x_{m:n}.$$

Saving surviving units can be used in future tests is one of the most important features to use T-II CS.

Competing risks models are vital tools in survival analysis, especially when dealing with multiple potential failure causes. These models become even more complex and insightful when combined with censoring, particularly Type-I censoring and Type-II censoring. The selected studies present a comprehensive exploration of various statistical approaches to analyzing lifetime data under different competing risks model with Type-I censoring and Type-II censoring, utilizing different probability distributions and methodological frameworks.

Starling *et al.* (2021) addresses the estimation of Weibull distribution parameters using a generalized Type-I censored data. The authors propose a novel approach utilizing modified SMOTE (Synthetic Minority Over-sampling Technique) to improve the accuracy of parameter estimates.

Almarashi and Abd-Elmougod (2020) investigated an accelerated model with competing risks based on Gompertz lifetime distributions using a type-II censoring. Their work focused on scenarios where the life tests are censored, meaning the study ends after a pre-set number of failures. The study highlights the applicability of the Gompertz distribution in modeling lifetimes under stress, especially in industrial reliability testing. The authors provided detailed derivations of the (MLE) and their asymptotic properties, which are crucial for the practical implementation of such models. Their results showed that the Gompertz model, combined with type-II censoring, offers robust and reliable estimations, making it a valuable tool in reliability engineering. n a similar vein, **Almarashi *et al.* (2020)** explored the Nadarajah and Haghighi (NH) distribution for statistical inference of competing risks lifetime data using type-II censoring. The NH distribution is known for its flexibility when modeling different shapes of hazard functions, which is particularly useful in survival analysis. This study provided a detailed examination of the Bayesian and classical estimation methods under the NH distribution, offering insights into the performance of these estimators under type-II censoring. The authors demonstrated that the NH distribution could effectively model lifetime data in the presence of competing risks, with Bayesian methods showing superiority in terms of interval estimation, especially when prior information is available. **Ramadan *et al.* (2023)** introduced the Akshaya distribution in the context of competing risks, applying it specifically to the progression from HIV infection to AIDS. Their study is significant as it extends the use of the Akshaya distribution, which is relatively new in survival analysis, to a real-world application under type-II censoring.The authors provided comprehensive statistical inference for the Akshaya distribution parameters, utilizing both frequentist and Bayesian approaches. The study's application to HIV data underscores the practical relevance of competing risks models in medical statistics, demonstrating how these models can be tailored to address specific public health challenges. Aljohani and Alfar (2020) focused on the Burr XII lifetime model under a step-stress partially accelerated life test (SSPALT) with type-II censoring. The Burr XII distribution is well-regarded for its versatility in modeling various life phenomena, and its application in SSPALT when using type-II censoring is particularly innovative. The experiment detailed the estimation procedures, including MLE and Bayesian estimators, under the SSPALT framework. Their results indicated that the Burr XII model, when used with type-II censoring in a step-stress test environment, could yield efficient and accurate estimations, making it a robust choice for accelerated life testing in reliability studies.

The type II censored competing risks model is:

Suppose there are k competing risks for units i let X_{i1}, X_{i2} be the potential failure times due to risks $k = \{1, 2\}$ respectively. The observed failure time T_i is $X_i = \min\{X_{i1}, X_{i2}\}$

Suppose n independent units are put on a life test, and before the start of the experiment the integer m is considered, let the first failure $X_{1;m}$ and its cause of failure is δ_1 be recorded. Then The second failure is $X_{2;m}$ and its cause of failure δ_2 are recorded and the study is continued until m th failure $X_{m;m}$ and its cause of failure δ_1 are recorded. The data $X_{1;m,\delta_1} < X_{2;m,\delta_2} < \cdots < X_{m;m,\delta_m}$ is type-II censored competing risk data. The likelihood function then is

$$L = \frac{n!}{(n-m)!}\left\{\bar{F}_1(x_m)\bar{F}_2(x_m)\right\}^{n-m} \times \prod_{i=1}^{m}\left[f_1(x_i)\left(\bar{F}_2(x_i)\right)\right]^{I(\delta_i=1)}\left[f_2(x_i)\left(\bar{F}_1(x_i)\right)\right]^{I(\delta_i=2)}, \quad [5]$$

Where $\bar{F}_i(.)$ is the survival (reliability) function and

$$I(\delta_i = j) = \{^{1,\ \delta_i = j}_{0,\ \delta_i \neq j},\ j = 1, 2,$$

for $0 < x_1 < x_2 < \cdots < x_m < \infty$.

Hybrid Censoring in Competing Risks Models

In a study published in Complexity**, Alghamdi (2021)** explored the use of a generalized Type-I hybrid censoring scheme within a partially accelerated model to analyze competing risks data from a Gompertz population. The research tackled the complexities of analyzing competing risks when the underlying distribution follows a Gompertz model. Alghamdi proposed a generalized Type-I hybrid censoring scheme that accommodates both the accelerated nature of life tests and the presence of competing risks. The study demonstrated that this scheme is particularly effective in improving the accuracy of parameter estimates in intricate reliability studies. The likelihood function is provided by equation [6] as follow:

$$L = \prod_{j=1}^{2} Q_j\{\prod_{i=1}^{v_i}\left[f_{1j}\left(x_{ij:n_j}\right)\bar{F}_{2j}\left(x_{ij:n_j}\right)\right]^{\delta_j}\left[f_{2j}\left(x_{ij:n_j}\right)\bar{F}_{1j}\left(x_{ij:n_j}\right)\right]^{I=\delta_j}$$

$$\times [\bar{F}_{1j}\left(x_{vj:n_j}\right)\bar{F}_{2j}\left(x_{vj:n_j}\right)]^{n_j-v_j} \quad [6]$$

Where

$$Q_j = \frac{n_j!}{(n_j-v_j)!} \quad [7]$$

Is a combinatorial coefficient that accounts for the different ways in which D uncensored events can be selected from n observations. $n-D$ represents the number of censored observations and

$$I(\delta_i = j) = \{^{1,\quad I=1}_{0,\quad I=2}$$

are indicator functions that take the value 1 when $\delta_i = 1, 2, 3$ and 0 otherwise

$f_1(x_i)$ and $f_2(x_i)$ are the probability density functions (PDFs) corresponding to the two different risk types.

$\bar{F}_1(x_i)$, $\bar{F}_2(x_i)$ are the survival function corresponding to the two different risk types.

Type-II hybrid censoring is an essential approach in the analysis of competing risks models. **Bai *et al.* (2020)** made notable advancements in this field by developing inference methods for an accelerated dependent competing risks model using the Marshall-Olkin bivariate Weibull distribution with non-constant parameters. Their study, published in the Journal of Computational and Applied Mathematics, tackled the challenges associated with dependent competing risks in accelerated life tests. By employing Type-II hybrid censoring, the authors enhanced the efficiency and precision of parameter estimation, especially in cases where failure times are only partially observed. Their findings underscored the effectiveness of Type-II hybrid censoring in managing dependent competing risks, providing a valuable tool for statisticians and engineers dealing with complex reliability data.

Abushal *et al.* (2022) investigated statistical inference techniques for partially observed causes of failure within the Lomax competing risks model. The paper focused on a generalized Type-II hybrid censoring scheme that integrates aspects of Type-II censoring with hybrid censoring methods to improve data analysis and inference. The likelihood function is:

$$L = QS(\eta)^{n-D} \prod_{i=1}^{D} \left[f_1(x_i)\left(\bar{F}_2(x_i)\right)\right]^{I(\delta_i=1)} \left[f_2(x_i)\left(\bar{F}_1(x_i)\right)\right]^{I(\delta_i=2)} \left[f(x_i)\right]^{I(\delta_i=3)}, i = 1,2,\ldots,D \quad [8]$$

where

$$Q = \frac{n!}{(n-D)!} \quad [9]$$

Is a combinatorial coefficient that accounts for the different ways in which D uncensored events can be selected from n observations. $n-D$ represents the number of censored observations.

$f_1(x_i)$ and $f_2(x_i)$ are the probability density functions (PDFs) corresponding to the two different risk types.

$\bar{F}_1(x_i), \bar{F}_2(x_i)$ are the survival function corresponding to the two different risk types. $I(\delta_i = 1), I(\delta_i = 2), I(\delta_i = 3)$ are indicator functions that take the value 1 when $\delta_i = 1,2,3$ and 0 otherwise

Unified hybrid censoring in Competing Risks Models

Unified hybrid censoring schemes represent a comprehensive approach to handling incomplete data in reliability and survival analysis, particularly when dealing with competing risks. These schemes combine features from multiple censoring methods, offering greater flexibility and efficiency in statistical inference. This review explores the key contributions to the field, focusing on Bayesian and non-Bayesian inference under the unified hybrid censoring. **Dutta and Kayal (2022)** conducted a seminal study on Bayesian and non-Bayesian analysis for the Weibull model using partially observed competing risks when using a unified hybrid censoring. Published in Quality and Reliability Engineering International, their research addresses the challenges of making accurate inferences when only partial data is available due to the presence of competing risks. The unified hybrid censoring scheme in this study integrates elements from Type-I censoring, Type-II censoring, and progressive censoring, providing a robust framework for statistical analysis. The authors demonstrated that their approach enhances the estimation accuracy for the Weibull distribution parameters, making it particularly useful in reliability engineering and life testing.

Expanding on the application of unified hybrid censoring, **Dutta *et al.* (2023)** explored inference for a family of inverted exponentiated distributions using the same censoring scheme. Their study, focuses on the versatility of the unified hybrid censoring approach in dealing with partially observed competing risks data. The authors developed both Bayesian and classical inference methods tailored to the inverted exponentiated distributions, demonstrating the effectiveness of their approach in a broader class of lifetime distributions. The study's findings underscore the flexibility of the unified hybrid censoring scheme in accommodating different distributional assumptions and its potential for broader applications in reliability analysis.

Lone and Panahi (2022) made a significant contribution to the literature by developing estimation procedures for a partially accelerated life test model based on a unified hybrid censored sample from the Gompertz distribution. Their study, published in Eksploatacja i Niezawodność, addressed the unique challenges associated with partially accelerated life tests, where testing conditions are deliberately altered to prompt failures. The unified hybrid censoring scheme utilized in this research offers greater flexibility in managing censored data, encompassing both time-censored and failure-censored observations. The authors introduced estimation procedures that enhance the reliability and accuracy of parameter estimates for the Gompertz distribution, a model extensively used in reliability analysis

$$L = \prod_{i=1}^{D} (h_1(x_i))^{D_1} (h_2(x_i))^{D_2} [S_1(x_i) S_2(x_i)] [S_1(T^*) S_2(T^*)]^{(n-D)} \quad [10]$$

Where $h_1(x_i)$ and $h_2(x_i)$ are the hazard functions for the two competing risks at time x_i These represent the instantaneous risk of the event happening at time x_i due to risk 1 and risk 2, respectively. D_1 and D_2 are indicators for the events caused by the first and second risks, respectively. $(h_1(x_i))^{D_1}$ and $(h_2(x_i))^{D_2}$ indicate that the likelihood depends on which risk is responsible for the event. $S_1(x_i)$ and $S_2(x_i)$ are the survival functions for the two risks at time x_i these represent the probability that an individual survives past time x_i without experiencing the event due to risk 1 or risk 2, respectively, and The product $\prod_{i=1}^{D}$ goes over all observed events i up to D, where D is the total number of events observed.

Progressive Censoring in Competing Risks Models

The Progressive Type II Censoring: let n independent units be placed in the life test at the same initial time $t_0=0$. The first failure $X_{1:m:n}$ is observed, R_1 remaining units from the surviving *n-1* units are withdrawn randomly from the test. Similarly, when the second failure $X_{2:m:n}$ is observed, R_2 surviving units from the remaining $n-2-r_1$ units are randomly withdrawn from the test. This process keep going until, the m^{th} m^{th} failure $X_{m:m:n}$ happen, all of the remaining units $n-m-\sum_{i=1}^{m} R_i$ are removed from the test . It is noted that an integer $m<n$ is determined, Prog CS $(R_1, R_2, ..., R_m)$ with $R_i > 0$ and $\sum_{i=1}^{m} R_i + m = n$ is specified before the experiment in the ProgT-II CS.

The joint probability density function of a ProgT-II CS sample $X_1 = X_{1:m:n}, X_2 = X_{2:m:n}, \dots, X_m = X_{m:m:n}$ with PDF $f(x)$ and CDF $F(x)$ can be expressed in the following form

$$f(x_1, x_2, ..., x_m) = A \prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \quad [11]$$

where x_i is used instead of $X_{i:m:n}$, R_i ?0 , $i = 1, 2, \dots, m$ and

$$A = n(n-1-R_1)(n-2-R_1-R_2)\dots\left(n-\sum_{i=1}^{m-1}(R_i+1)\right). \quad [12]$$

Progressively Type-I censoring is a powerful statistical method used in survival analysis, especially when dealing with accelerated life tests and competing risks models. This censoring scheme involves removing units from a life test at pre-specified times, providing a more flexible approach to handling censored data.

Bai *et al.* (2022) conducted an influential study on the application of Type-I progressive censoring in the context of step-stress accelerated life tests (SSALT) with dependent competing risks. Their work, published in Communications in Statistics-Theory and Methods, represents a significant contribution to the field of reliability analysis. The Type-I progressive censoring scheme used in this study provides flexibility in life testing, allowing for the adjustment of test conditions based on observed data. This adaptability is particularly useful in accelerated life tests, where stress levels change over time. By incorporating dependent competing risks, the study addresses a common challenge in reliability analysis, where different failure modes are not independent. The use of copula functions to model dependencies adds robustness to the statistical inference. The methods developed by Bai, Shi, and Ng have practical relevance in industries where accelerated life testing is common, such as electronics, automotive, and materials engineering. The ability to make accurate inferences about product reliability under varying stress conditions is crucial for improving product design and quality assurance. Progressive Type-II censoring is an advanced statistical technique widely used in survival analysis, particularly when dealing with competing risks and reliability models. In this scheme, the number of failures is fixed, and the removal of remaining units occurs progressively throughout the test. Progressive Type-II censoring schemes provide a flexible and efficient way to handle censored data, especially when the exact time of censoring is unknown. The cited studies represent significant contributions to the fields of reliability analysis and survival modeling, offering practical solutions for industries ranging from engineering to biomedical research.

These references highlight the diversity of applications and methodological advancements in the study of progressive Type-II censoring under competing risks, emphasizing its importance in modern statistical analysis.

Ren and Gui (2021a) this study, published in Computational Statistics, explores the analysis and best censoring schemes for a progressively Type-II censored model with competing risks applied to the generalized Rayleigh distribution. The authors provide comprehensive methods for parameter estimation and optimal censoring strategies, which are crucial for efficient and cost-effective reliability testing.

Abushal *et al.* (2022) this research, featured in Complexity, deals with the estimation of the Akshaya failure model using a progressive censoring scheme. The study includes an application to the analysis of thymic lymphoma in mice, demonstrating the practical implications of the model in biomedical research.

Moharib Alsarray *et al.* (2023) published a study in the Journal of Applied **Statistics** that focuses on monitoring the Weibull shape parameter in the context of independent competing risks under a progressive censoring scheme. This research is particularly relevant to industries where the Weibull distribution is frequently used to model lifetimes.

Ahmed *et al.* (2020) explored inference methods for progressively Type-II censored competing risks data from the Chen distribution. Published in the Journal of Applied Statistics**,** the study offers both theoretical insights and practical applications, showcasing the versatility of the Chen distribution in modeling competing risks.

Qin and Gui (2020), in their study featured in the Journal of Computational and Applied Mathematics, examined statistical inference for the Burr-XII distribution under progressive Type-II censored competing risks data with binomial removals. The authors provide new perspectives on handling complex censoring mechanisms in reliability data.

Mondal and Kundu (2020) focused on inference for Weibull parameters under a balanced two-sample Type-II progressive censoring scheme in their study published in Quality and Reliability Engineering International. This research is especially relevant for comparative studies in reliability engineering.

Alam and Ahmed (2022) published a study in the Journal of Statistical Computation and Simulation that investigates inference on maintenance service policy under step-stress partially accelerated life tests using progressive censoring. The research offers practical applications for maintenance and warranty analysis in engineering.

El-Sherpieny *et al.* (2023) presented a study in Sankhya A on Bayesian and non-Bayesian estimation for the parameters of the bivariate generalized Rayleigh distribution based on Clayton copula under progressive Type-II censoring with random removal. This research contributes to the field of multivariate survival analysis.

Dey *et al.* (2022) investigated inference on the Nadarajah-Haghighi distribution under constant stress partially accelerated life tests with progressive Type-II censoring in their study published in the Journal of Applied Statistics**.** This research is crucial for understanding the impact of stress on product lifetimes in reliability studies. The likelihood function for the progressive Type-II censoring with competing risks model is:

$$L = A\prod_{j=1}^{2}\prod_{i=1}^{m_j}\left[f_j(x_i)\left(\bar{F}_{3-j}(x_i)\right)\right]^{I(\delta_i=j)}\left[\bar{F}_j(x_i)\right]^{R_i} \quad [13]$$

where

$$A = n\prod_{i=1}^{m-1}\left[n-\sum_{k=1}^{i}(R_k+1)\right] \quad [14]$$

Adaptive Progressive Type-II Censoring in Competing Risks Models

The Adaptive Progressive Type-II Censoring scheme is a type of censoring scheme in which $_m$ is fixed and the Prog *CS* $(R, R_2, \dots R_m)$ is specified before the start of the experiment. Let n independent units be placed in the life test at the same initial time $T_0=0$. The experimenter set a time *T*, *T*, that can be considered as an ideal total test, but the study time is allowed to exceed time *T*.

When the first failure $X_{1:m:n}$ is observed, R_1 remaining units from the surviving $_{n-1}$ units are removed randomly from the test. Similarly, when the second failure $X_{2:m:n}$ happen, R_2 remaining units from the surviving $n-2-R_1$ units are randomly removed from the test. This process remain until the m^{th} failure $X_{m:m:n}$ is observed, all of the remaining units $n-m-\sum_{i=1}^{m}R_i$ are removed randomly from the test and it will have a usual ProgT-II CS with the prefixed Prog CS. Otherwise, when the study time passes the set time T but the number of observed failures haven't reached m, then we adopt the number of units progressively withdrawn from the study upon failure by setting $R_{J+1}, R_{J+2}, \dots, R_{m-1}=0$ and $R_m = n-m-\sum_{i=1}^{J}R_i$

Where $X_{J:m:n}<T<X_{J+1:m:n}$, and $X_{J:m:n}$ is the J^{th}failure happen before the set time T and $J+1<m$. Hence, the effectively applied scheme is $R_1, R_2, \dots, R_J, 0, 0, \dots, 0, n-m-\sum_{i=1}^{J}R_i$.

This formula on leads to the end of the experiment as soon as possible if the

$(J+1)^{th}$ failure time is greater than T. The total test time will not be too far away from the time T.An AT-II Prog CS can be reduced to well-known types of censoring scheme as the following extreme cases:

1) If $T \to \infty$, then AT-II Prog CS reduces to the ProgT-II CS.
2) If $T = 0$, then AT-II Prog CS reduces to the T-II CS.

The likelihood function for given $J = j$ is given by

$$L(\lambda \mid J=j)=d_j \prod_{i=1}^{m}\left[f\left(x_{i;m,n}\right)\right]\prod_{i=1}^{j}\left[1-F\left(x_{i;m,n}\right)\right]^{R_i} \times \left[1-F\left(x_{m;m,n}\right)\right]^{n-m-\sum_{i=1}^{j}R_i},$$

$0 < x_{1;m,n} < x_{2;m,n} < \cdots < x_{m;m,n} < \infty$ [15]

where

$$d_j = \prod_{i=1}^{m}\left[n-i+1-\sum_{k=1}^{\max\{i-1,\, j\}} R_k\right]. [16]$$

Adaptive progressive Type-II censoring is an advanced statistical technique that offers flexibility in the censoring process, allowing adjustments based on the observed data. This method is particularly useful in survival analysis, reliability engineering, and other fields where the timing of failures is critical. Adaptive progressive Type-II censoring schemes allow for more flexible and responsive data collection processes, making them ideal for situations where the exact timing of failures cannot be predetermined. The ability to adaptively censor data based on observed outcomes improves the efficiency of statistical analyses and provides more accurate parameter estimates, particularly in complex models like competing risks and accelerated life testing.

Nassar *et al.* (2022) this study, published in Mathematical Problems in Engineering, explores estimation methods for adaptive progressively censored data under competing risks models, with applications in engineering. The authors provide practical solutions for dealing with complex censoring schemes in real-world engineering problems, highlighting the adaptability of the method.

Ren and Gui (2021b) published in Applied Mathematical Modelling, this paper focuses on the statistical analysis of adaptive Type-II progressively censored data within the context of Weibull models. The study offers comprehensive methods for parameter estimation and model fitting, demonstrating the effectiveness of adaptive censoring in handling Weibull-distributed lifetimes.

Almalki *et al.* (2022) this research, featured in Alexandria Engineering Journal, investigates a partially constant-stress accelerated life test model for the Kumaraswamy distribution under adaptive Type-II progressive censoring. The study is particularly relevant for reliability testing and provides insights into the accelerated life testing framework.

Dutta *et al.* (2024) this forthcoming paper, to be published in Computational Statistics, explores Bayesian reliability analysis for the logistic exponential distribution using adaptive progressive Type-II censored data. The study offers a robust Bayesian approach to handling censoring in survival data, with potential applications in medical and reliability studies.

The number of units is $m(m<n)$ and the censoring scheme is type-II progressive $\Re = (R_1, R_2, \ldots R_m)$ are determined, and $R_i > 0,\ i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} R_i = n + m.$

$X_{1:n}$ is observed as the first failure then $X_{1:m:n} = X_{1:n}$.Then R_1 of the $n \not\leq 1$ units that are still surviving are chosen randomly and removed from the experiment. Then, the second failure is observed $(X_{2:m:n})$ as the unit with the smallest life time, as a result R_2 of the remaining $n - R_1 - 2$ units are removed from the experiment, and this procedure is going to be repeated until all of the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ units are censored from the experiment at the time of m^{th} observed failure $X_{m:m:n}$. Then the progressively type-II censoring data are given as follow $\left(X_{1:m:n}, \delta_1^*\right), \left(X_{2:m:n}, \delta_2^*\right), \cdots, \left(X_{n:m:n}, \delta_n^*\right),$ the notation * is introduced since δ_i^* are concomitants of the order statistics hence, δ_i^* is not equal to δ_i.

Let $\delta_i^* = k$, $k = 1, 2$ where δ_i^* indicates that the failure of units at time $X_{i:m:n}$ was failed by the cause of failure k. Let $I_k(A)$ be the indicator of the event A, where $I_1\left(\delta_i^* = 1\right) = 1$ $\delta_i^* = 1$ and $I_1\left(\delta_i^* = 1\right) = 0$ where $\delta_i^* \neq 1$ and $I_2\left(\delta_i^* = 2\right) = 1$ where $\delta_i^* = 2$ and $I_2\left(\delta_i^* = 2\right) = 0$ where $\delta_i^* \neq 2$. the number of failure are attributed to the first and the second cause of failure, and then described by random variables $m_{1=} \sum_{i=1}^{m} I\left(\delta_i^* = 1\right)$ and $m_{2=} \sum_{i=1}^{m} I\left(\delta_i^* = 2\right)$ where $m_1 + m_2 = m$ and $m > 0$.

Let T be the test predetermined expected completion time, but the test time is allowed to exceed it. If this condition is not fellfield the test ends when $X_{j:m:n} < T < X_{j+1:m:n}$, where $1 \le j \le m-1$. No survival unit is going to be discarded, that means the censoring scheme $\Re$ will be changed to $R_{j+1} = R_{J+2} = \cdots = R_{m-1} = 0,$

$R_m = n - m - \sum_{j=1}^{J} R_j$. Where $J = \max\left\{j : X_{j:m:n} < T\right\}$. And then in the presence of competing risks data and under the adaptive type-II progressive censoring scheme, the observation takes the form

$$\left(X_{1:m:n}, \delta_1^*, R_1\right), \left(X_{2:m:n}, \delta_2^*, R_2\right), \ldots, \left(X_{j:m:n}, \delta_j^*, R_j\right),$$
$$\left(X_{j+1:m:n}, \delta_{j+1}^*, 0\right), \ldots, \left(X_{m-1:m:n}, \delta_{m-1}^*, 0\right), \left(X_{m:m:n}, \delta_{m-1}^*, R_m\right)$$

$$L(\theta; x) = C_j \prod_{i=1}^{m} \left[f_1(x_i)\left(\bar{F}_2(x_i)\right)\right]^{I(\delta_i = 1)} \left[f_2(x_i)\left(\bar{F}_1(x_i)\right)\right]^{I(\delta_i = 2)} \prod_{i=1}^{J} \left[\bar{F}(x_i)\right]^{R_i} \left[\bar{F}(x_m)\right]^{R^*} \quad [17]$$

where

$$C_j = \prod_{i=1}^{m} \left(n - i + 1 - \sum_{j=1}^{\max\{i-1, J\}} R_j,\right), R^* = n - m - \sum_{i=1}^{J} R_i, \text{ and } \bar{F}_k(x_i) = 1 - F_k(x_i),\ k = 1, 2. \quad [18]$$

Improved Adaptive Progressive Type II in Competing Risks Models

Improved adaptive progressive Type-II censoring scheme [IAT-II PCS] enhances the flexibility and efficiency of the traditional adaptive progressive Type-II censoring methods by incorporating additional strategies for data collection and analysis. The Improved adaptive progressive Type-II censoring is described as follows:

let $D_2 = \begin{cases} m, case\ I\ and\ case\ II \\ k_2, case\ III \end{cases}$ which denotes number of failures before T_2. Then the IAT-II PCS sample is $X_{1:m:n}$, $X_{2:m:n}$, …, $X_{D2:m:n}$ and $D_1 = \begin{cases} m, case\ I \\ k_1, case\ II\ and\ case\ III \end{cases}$ The progressive censoring scheme is R. Then, the corresponding likelihood function based on IAT-IIPCS can be expressed as:

$$L = C\left[\prod_{i=1}^{D_2} f\left(X_{1:m:n}\right)\right]\left[\prod_{i=1}^{D_1}\left(1 - F\left(X_{1:m:n}\right)\right)^{R_i}\right].\left[1 - F\left(T^*\right)\right]^{B} \quad [19]$$

where

$$C = \begin{cases} \prod_{i=1}^{m}(n - i + 1 - \sum_{s=1}^{i-1} R_s\ , for\ case\ 1 \\ \prod_{i=1}^{m}(n - i + 1 - \sum_{s=1}^{k^1} R_s\ , for\ case 2 \\ \prod_{i=1}^{k^2}(n - i + 1 - \sum_{s=1}^{i-1} R_s\ , for\ case 3 \end{cases} \quad [20]$$

and

$$B = \left(B_1, B_2, B_3\right) \text{ where} B_1 = 0, \text{ for case I}, B_2 = n - m - \sum_{i=1}^{k_1} R_i, \text{ for case II},$$
$$B_3 = n - k_2 - \sum_{i=1}^{k_1} R_i, \text{ for case III}.$$

This approach allows for more precise handling of partially observed failure causes and optimizes the censoring process to better accommodate varying types of data and model requirements. The improved adaptive progressive Type-II censoring schemes offer several advantages over traditional methods:

- Enhanced Flexibility: These schemes adapt to varying data conditions and partially observed failure causes, providing a more robust framework for statistical analysis.
- Optimized Censoring: By incorporating improved strategies, the methods ensure more efficient use of data and better handling of complex models.
- Increased Accuracy: The enhanced techniques lead to more precise parameter estimates and reliable model inference, particularly in scenarios involving competing risks and complex distributions.

Dutta and Kayal (2023) this paper, addresses the inference of competing risks models when failure causes are only partially observed. The study introduces improved adaptive Type-II progressive censoring techniques to handle such complex scenarios effectively.

Elshahhat and Nassar (2024) featured in Statistical Papers, this research focuses on the inference of improved adaptive progressively censored data within the context of Weibull lifetime models. The authors provide methodologies for accurate parameter estimation and model fitting under enhanced adaptive censoring schemes.

Yan and Yu (2021) this research, provides statistical inference techniques for the reliability of Burr-XII distributions using improved adaptive Type-II progressive censoring. The paper demonstrates the application of enhanced censoring methods to Burr-XII models.

Adaptive Type-I Progressive Hybrid Censoring in Competing Risks Models

Adaptive Type-I progressive hybrid censoring [AT-I PHC] is a sophisticated censoring method designed to improve the efficiency and accuracy of statistical inferences in reliability and survival analysis. This method combines elements of Type-I, progressive, and hybrid censoring schemes, allowing for a more flexible and adaptive approach to dealing with incomplete data, particularly when competing risks are present see, **Nassar and Dobbah (2020)**. Description of the AT-I PHC Scheme:

Initial Setup: n units are placed in the study with a censoring scheme $R_1, R_2, \ldots, R_m$ .The experiment is predetermined to end at time T, where T is a fixed point in time $T \in (0, \infty)$.

Censoring Process:

At the first failure, R_1 units are removed randomly from the surviving units.

At the second failure, R_2 units are removed randomly, and so on.

Let J denote the number of failures that happen before the predetermined time T. Hence we have two Possible Scenarios:

1-If the m^{th}failure happen before the set time T:

After the m^{th} failure, no further units are removed, and the process continues to observe failures until time T. At T time the surviving $R_J^* = n - J - \sum_{i=1}^{J} R_i$ units are withdrawn, and the study is ended.

2- If the time T happen before the occurrence of the m^{th} failure:

The study then follows a censoring scheme with $R_1, R_2, \ldots, R_J$ up to the set time T.

Given the AT-I PHC scheme, the maximum likelihood L is given by:

$$L = C_J \prod_{i=1}^{J} f\left(x_{i\,:\,J\,:\,n}\right) \times \left[1 - F\left(x_{i\,:\,J\,:\,n}\right)\right]^{R_i} S(T)^{R_J^*} . [21]$$

Where

$$C_J = \prod_{i=1}^{J} \varphi_i \, [22]$$

With $\varphi_i = m - i + 1 + \sum_{j=1}^{m} R_j$ and $R_J^* = n - j - \sum_{i=1}^{j} R_i$

The following review discusses a study that contributed to the development and application of this censoring scheme when competing risks are present.

Okasha and Mustafa (2020) explored the application of adaptive Type-I progressive hybrid censoring in the context of competing risks models, focusing on E-Bayesian estimation for the Weibull distribution. Their study, published in Entropy, addresses the challenges of making accurate parameter estimates in reliability analysis when dealing with censored data. The adaptive nature of the censoring scheme allows for adjustments based on the observed data, improving the efficiency of the estimation process. The authors developed E-Bayesian estimation methods, which provide a more flexible and robust approach compared to traditional Bayesian estimation. Their results demonstrate that the adaptive Type-I progressive hybrid censoring scheme, combined with E-Bayesian estimation, offers significant improvements in the accuracy of parameter estimates for the Weibull distribution, making it particularly useful in reliability and life testing applications.

The likelihood using the adaptive type-I progressive censoring and when competing risks data are percent is given by:

$$L = C\prod_{i=1}^{J}\left\{\left[f_1(x_i)\bar{F}_2(x_i)\right]^{I(C_1=1)}\left[f_2(x_i)\bar{F}_1(x_i)\right]^{I(C_1=2)}\left[\bar{F}_1(x_i)\bar{F}_2(x_i)\right]^{R_i}\right\}\left[\bar{F}_1(\tau)\bar{F}_2(\tau)\right]^{R_J^*}. \quad [23]$$

Where $x_i = x_{i\,:\,m\,:\,n}$, C is a constant.

Adaptive Type-II Progressive Hybrid Censoring in Competing Risks

In the last few years, the ProgT-II CS was a flexible type of censoring used in many lifetime experiments and reliability analysis, but a demerit of ProgT-II CS is that the length of the study can be very long if the units under test are reliable. For this reason, the (T-II ProgH CS) was suggested: Let n independent units be put to the life test at the same initial time $t_{0=0}$. The experimenter set a time T and m is a number of failure units. At the first failure $Y_{1:m:n}$, R_1 surviving units from $n-1$ units are removed randomly. Similarly at the second failure $Y_{2:m:n}$, R_2 surviving units from the remaining $n-1-R_1$ units are removed randomly and so on. If the m^{th} failure $Y_{m\,:\,m\,:\,n}$ happen before the time T, the study end at the failure time $Y_{m\,:\,m\,:\,n}$ and all of the remaining surviving units $n-m-\sum_{i=1}^{m} R_i$ are removed. However, if the m^{th} failure does not happen before the set time T and only the J failures happen before the set time T, where $0 < J < m$; then, at the time T all the surviving R_J^* units are withdrawn and the study is terminates at the time T. Note that $R_J^* = n - (R_1 + R_2 + \ldots + R_J) - J$.

Hence, in the presence of T-II ProgH CS, the following types of observations are percent:

$$Case\,I\;:\;\left\{Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{m:m:n}\right\} \quad if \quad Y_{m:m:n} < T,$$

or

$$Case\,II\;:\;\left\{Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{J:m:n}\right\} \quad if \quad Y_{J:m:n} < T < Y_{J+1:m:n}.$$

It is clear that from Case II, $Y_{J\,:\,m\,:\,n} < T < Y_{J+1\,:\,m\,:\,n} < \ldots < Y_{m\,:\,m\,:\,n}$ and $Y_{J+1\,:\,m\,:\,n}$,..., $Y_{m:m:n}$ are not observed. The likelihood function of distribution parameter λ for Case I of T-II ProgH CS can be written as

$$L(\lambda) = B_1\prod_{i=1}^{m} f\left(y_{i\,:\,m\,:\,n}\right)\left[1 - F\left(y_{i\,:\,m\,:\,n}\right)\right]^{R_i}, \quad [24]$$

and for Case II

$$L(\lambda) = B_2\left\{\prod_{i=1}^{J} f\left(y_{i\,:\,m\,:\,n}\right)\left[1 - F\left(y_{i\,:\,m\,:\,n}\right)\right]^{R_i}\right\}\times\left[1 - F(T)\right]^{R_J^*}, \quad [25]$$

where B_1 and B_2 are constants

$$B_1 = \prod_{i=1}^{m}\left[n - \sum_{l=1}^{i-1}(1+R_l)\right] \text{ and } B_2 = \prod_{i=1}^{J}\left[n - \sum_{l=1}^{i-1}(1+R_l)\right].$$ [26]

Adaptive Type-II progressive hybrid censoring combines features of both progressive and hybrid censoring schemes, enhancing the ability to handle complex data scenarios involving competing risks. This approach integrates adaptive mechanisms to optimize the censoring process and improve parameter estimation. Adaptive Type-II progressive hybrid censoring combines elements from various censoring schemes to address complex data analysis scenarios:

- Enhanced Adaptability: Integrates adaptive features to improve the handling of diverse and partially observed data.
- Optimized Data Utilization: Employs hybrid strategies to maximize the efficiency and accuracy of data collection and analysis.
- Advanced Inference Techniques: Provides robust methods for parameter estimation and model fitting in the presence of competing risks and complex distributions.

Nassr *et al.* (2021) this paper, published in Thailand Statistician, explores the statistical inference techniques for the extended Weibull distribution using adaptive Type-II progressive hybrid censored competing risks data. The study provides methodologies for effective data analysis and parameter estimation.

Du and Gui (2022) featured in Journal of Applied Statistics, this research focuses on the statistical inference of data with dependent competing risks modeled by the bivariate exponential distribution. The paper employs adaptive Type-II progressive hybrid censoring to enhance the accuracy of the analysis.

Liu and Gui (2020) published in Mathematics, this study presents methods for estimating parameters of the two-parameter Rayleigh distribution under adaptive Type-II progressive hybrid censoring. The authors address challenges related to competing risks and data handling.

Vardani *et al.* (2024) featured in Physica Scripta, this research investigates statistical inference techniques for the Marshall-Olkin bivariate Kumaraswamy distribution under adaptive progressive hybrid censoring with dependent competing risks data.

The joint likelihood function of the adaptive Type-II progressive hybrid censored data.

The likelihood function of the adaptive Type-II progressive hybrid censoring

$$L = C\prod_{i=1}^{m}\left[f_1(x_i)\overline{F}_2(x_i)\right]^{I(\delta_i=1)}\left[f_2(x_i)\overline{F}_1(x_i)\right]^{I(\delta_i=2)}\left[f_1(x_i)\overline{F}_2(x_i)+f_2(x_i)\overline{F}_1(x_i)\right]^{I(\delta_i=*)}$$ [27]
$$\times\prod_{i=1}^{D}\left[\overline{F}_1(x_i)\overline{F}_2(x_i)\right]^{R_i}\left[\overline{F}_1(x_m)\overline{F}_2(x_m)\right]^{R^*},$$

where $m = m_1 + m_2$, $f_k(x_i)$ is the pdf, $\overline{F}_k(x_i)$ is the survival function, $k = 1, 2$

Generalized progressive hybrid censoring in Competing Risks Models

Generalized progressive hybrid censoring integrates various censoring schemes to enhance the analysis of competing risks and lifetime data. This approach offers a comprehensive framework for handling partially observed data and optimizing inference procedures. Generalized progressive hybrid censoring enhances traditional censoring schemes by combining multiple methods:

- Flexible Data Handling: Allows for more versatile data analysis by integrating different censoring strategies.
- Enhanced Inference: Improves accuracy and reliability of parameter estimation in complex models with competing risks.
- Broader Applications: Applicable to various distributions and practical scenarios, including engineering and reliability studies.

Mahto *et al.* (2022) this paper, published in Journal of Applied Statistics, provides methods for inference in competing risks models involving Kumaraswamy distribution under generalized progressive hybrid censoring.

Cho and Lee (2021) featured in Symmetry, this study focuses on exact likelihood inference for exponential data under generalized Type II progressive hybrid censoring within competing risks models.

Singh *et al.* (2021) oublished in Quality and Reliability Engineering International, this research examines inference techniques for the two-parameter Rayleigh distribution in the context of generalized progressive hybrid censoring.

Wang *et al.* (2020) this paper, in Journal of Computational and Applied Mathematics, discusses methods for analyzing Weibull competing risks data with generalized progressive hybrid censoring.

Lodhi and Wang (2021) featured in Journal of Statistical Computation and Simulation, this study explores inference methods for various inverted exponentiated distributions under generalized progressive hybrid censoring.

Wang and Li (2020) published in Communications in Statistics-Simulation and Computation, this research investigates exponential data under generalized progressive hybrid censoring.

The likelihood function

$$l = \begin{cases} C_1 \prod_{i=1}^{k}\left[h_{c_i}(y_{i:m:n})\prod_{j=1}^{2} s_j(y_{i:m:n})\right]^{\delta_i}\left[f(y_{i:m:n})\right]^{1-\delta_i}\left[S(y_{i:m:n})\right]^{R_i}, & \text{for case I,} \\ C_2 \prod_{i=1}^{d}\left[h_{c_i}(y_{i:m:n})\prod_{j=1}^{2} s_j(y_{i:m:n})\right]^{\delta_i}\left[f(y_{i:m:n})\right]^{1-\delta_i}\left[S(y_{i:m:n})\right]^{R_i}\left[S(T)\right]^{R^*_{d+1}}, & \text{for case II} \\ C_3 \prod_{i=1}^{m}\left[h_{c_i}(y_{i:m:n})\prod_{j=1}^{2} s_j(y_{i:m:n})\right]^{\delta_i}\left[f(y_{i:m:n})\right]^{1-\delta_i}\left[S(y_{i:m:n})\right]^{R_i}, & \text{for case III} \end{cases} \quad [28]$$

where $R^*_{d+1} = n - \sum_{i=1}^{d}(R_i + 1)$ [29] and

$$C_1 = \prod_{j=1}^{k}\sum_{s=j}^{m}(R_s + 1), \quad \text{case I},$$

$$C_2 = \prod_{j=1}^{d}\sum_{s=j}^{m}(R_s + 1), \quad \text{case II},$$

$$C_3 = \prod_{j=1}^{m}\sum_{s=j}^{m}(R_s + 1), \quad \text{case III}.$$

Joint and Interval Censoring in Competing Risks Models

Progressive interval censoring, is a type of censoring where the failure times are only observed within intervals, and the intervals themselves are progressively censored.

Azizi *et al.* (2020) addressed the inference for competing risks models when the data are progressively interval censored. The focus is on Weibull lifetime distributions, which are commonly used to model reliability and survival data. the methodology is applicable to real-world scenarios where data collection is limited to intervals, which is common in medical studies and reliability testing. This work extends the understanding of how to handle interval-censored data within the competing risks framework, providing tools for more accurate estimation and prediction in such settings.

Consider a scenario where the data are progressively interval censored at predetermined times $s_1, s_2, \ldots, s_k$ where $0 < s_1 < s_2 < \ldots < s_k$. At the i^{th} interval, we observe the pairs (n_{ij}, r_i) for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, s$

Where n_{ij} represents the number of failures in the interval $(s_{i-1}, s_i]$ at the j^{th} risk.

r_i represents the number of remaining units that are removed from the test at time s_i.

m_i is the number of non-removed remaining units at the start of the i^{th} stage.

Assume that the number of failures at each interval follows a multinomial distribution given by:

$$\left(n_{i1}, n_{i2}, \cdots, n_{is}\right) | \left(n_{i-1}, n_{i-2}, \cdots, n_{i-s}, ..., n_{11}, n_{12}, \cdots, n_{1s}, r_{i-1}, ..., r_i\right)$$

$$\sim Multinomial\left(m_i, q_{i1}, q_{i2}, ..., q_{is}, 1-q_i\right)$$

$\lambda_j\left(1-e^{-\lambda_j(s_i - s_{i-1})}\right)$ is the probability of failure in the interval $(s_{i-1}, s_i]$ due to the i^{th} cause of failure,

where

$$q_{ij} = \frac{F(\tau_i, j) - F(\tau_{i-1}, j)}{1 - F(\tau_{i-1}, j)}, \text{ [30]}$$

and

$$q_i = \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})}, \text{[31]}$$

Moreover, the removal of units at the i^{th} stage is assumed to follow a binomial distribution:

$$r_i \mid n_{i1}, \cdots, n_{is}, \cdots, n_{11}, \cdots, n_{1s}, r_{i-1}, \cdots, r_1 \sim binomial\left(m_i - n_{i+}, p_i\right),$$

where $n_{i+} = \sum_{j=1}^{s} n_{ij}$ is the total number of failures at the i^{th} stage and p_i is the probability that a unit to be removed from the test at the i^{th} stage.

The likelihood function of the competing risks model with a progressive interval censoring is given by:

$$L \propto \prod_{i=1}^{k} f\left(n_{i1}, \cdots, n_{is}, \cdots, n_{i-1.1}, \cdots, n_{i-1,s}, \cdots, n_{11}, \cdots, n_{1s}, r_{i-1}, \cdots, r_1\right)$$
$$\times f\left(r_i \mid n_{i1}, \cdots, n_{is}, \cdots, n_{11}, \cdots, n_{1s}, r_{i-1}, \cdots, r_1\right)$$
$$\propto \prod_{i=1}^{k} \prod_{j=1}^{s} q_{ij}^{n_{ij}} \left(1-q_i\right)^{m_i - n_{i+}}$$
$$= \text{constant} + \prod_{i=1}^{k} \prod_{j=1}^{s} \frac{\lambda_j}{\lambda^*} \left(\frac{q_i}{1-q_i}\right)^{n_{ij}} \left(1-q_i\right)^{m_i}, \text{[32]}$$

Joint censoring combines features of Type-1 censoring, where observations are censored at a fixed time, with the additional complexity of competing risks. The balanced joint progressive Type-II censoring scheme involves censoring data in a way that allows for effective comparison between populations.

Conclusion

This paper provides a comprehensive review of censoring schemes in the context of competing risks models, emphasizing their critical role in survival analysis across a variety of disciplines. By exploring Type-I, Type-II, hybrid, and progressive censoring schemes, we have highlighted the strengths and limitations of each approach, particularly in handling incomplete data where multiple failure causes exist. The integration of censoring methods with competing risks models allows for more robust and accurate estimation of key parameters, such as survival functions and hazard rates.

Recent advancements in statistical methodologies, including both Bayesian and classical inference, have significantly improved the application of these models, particularly in reliability engineering, clinical trials, and industrial testing. The flexibility of modern censoring schemes, such as progressive and hybrid censoring, offers enhanced data utilization, making them powerful tools in handling complex survival data.

This review underscores the importance of censoring schemes in improving the accuracy of reliability and risk analysis. Future research should focus on refining these methods further, particularly in real-world applications, to address emerging challenges in medical statistics, engineering, and other fields where competing risks and incomplete data are prevalent.

Abushal *et al.* (2021) aimed to provide statistical inference for Burr XII lifetime models under joint Type-1 censoring in the context of competing risks. The study likely explored statistical inference methods for Burr XII distributions under joint censoring, focusing on how to handle data censored due to multiple competing risks. While the paper was retracted, it initially aimed to contribute to the understanding of Burr XII lifetime models under joint censoring, a challenging area in survival analysis.

The balanced joint Type-II progressive censoring scheme involves censoring data progressively and balancing the censoring to manage the risks effectively.

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